Burgers turbulence with random forcing: Similarity functional solution of the Hopf equation

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(Received 25 July 1994; revised manuscript received 21 December 1994)

For the problem of Burgers turbulence with random forcing, a similarity functional solution of the Hopf equation is presented and compared with scaling arguments and replica Bethe-ansatz treatments. The corresponding field theory is almost nonanomalous. In one dimension the local fluctuations develop self-similar time-dependent behavior, while relative fluctuations within the correlation length form a steady state with Gaussian distribution. This is the precise meaning of the so-called fluctuation-dissipation theorem. The one-dimensional properties are also studied numerically. It is shown that the fluctuation-dissipation theorem is invalid above one dimension and higher-order cumulants are nonzero. In two dimensions the cumulants exhibit a logarithmic spatial dependence, which is close to but different from that in the Edwards-Wilkinson case. No other similarity functional solution is found, which may indicate that the "strong-coupling" results are not described by the forced Burgers equation.

PACS number(s): 47.10.+g, 05.40.+j, 68.10.Jy

I. INTRODUCTION

Burgers model of turbulence [1,2], interface growth [3], ballistic deposition [3,4], domain walls in the anisotropic random-bond Ising model [5,6], directed polymers [7], and other phenomena is described by the Burgers equation with external conservative stirring

$$\partial_t \mathbf{v} = \nu \nabla^2 \mathbf{v} - \tfrac{1}{2} \nabla v^2 + \nabla \eta \ , \qquad (1.1)$$

which with the help of the velocity potential $\nabla h = -\lambda^{-1} \mathbf{v}$ gives an equation

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta \ , \qquad (1.2)$$

also known as the Kardar-Parisi-Zhang equation [3].

In our previous papers I and II [8] we considered the problem of random initial conditions and developed a Liouville-like field theory which enables one to calculate correlators of the field and study their time relaxation. It was shown that three different classes of initial conditions decay unlike each other and universality is limited. In this paper we address the development of Burgers turbulence starting with the Burgers field at rest $\mathbf{v}(\mathbf{x}, t=0)=0$ [or $h(\mathbf{x},0)=0$] under the action of external conservative stirring $\nabla \eta$ which is distributed as a Gaussian uncorrelated noise in space and time,

$$Q[\eta] \sim \exp\left[-\frac{1}{4D} \int \eta^2 dx\, dt \right] \ , \qquad (1.3)$$

and subsequently make a generalization for a non-Gaussian noise. A spatial cutoff a is required in all dimensions to keep the problem well defined.

Noise introduces spatial correlations and forms a distribution of the velocity field with steady-state features at length scales of the correlation length $L(t)$. At larger distances the velocity remains uncorrelated. The situation is somewhat analogous to the physics of critical phenomena, say, ϕ^4 theory [9], provided that the role of time is played by $T - T_c$. However, the exponents of the $d=1$ version of Burgers turbulence are simple fractions, $\frac{1}{3}$ and $\frac{2}{3}$ [1–3,6] (to be properly presented below). This provides some hope that in the associated field theory the anomaly is limited. Let us recall that in critical phenomena it was the anomalous behavior of correlators with merging points, first indicated in [10,11], that resulted in too general similarity functionals (also known as "bootstraps") for the hierarchy of correlators. As a result additional unknown constraints were needed. This method was subsequently in decline due to the invention of the ϵ expansion and conformal field theory. Seemingly nonanomalous exponents of the problem (1.1)–(1.3) suggest that the situation here may be different. Note that nonanomalous does not mean trivial; they are not necessarily diffusionlike exponents.

In Sec. II we present the similarity functional for the Hopf equation associated with (1.1)–(1.3). Kardar's replica Bethe-ansatz results [6] are rederived in Sec. III to help build the explicit similarity functional in Sec. IV. As explained in Secs. III and IV, the similarity functional contains correlators which are not present in the results obtained by using the ground state wave function of the replicated Hamiltonian. The large time limit and the replica limit do not commute, and that is why replica Bethe-ansatz results are incomplete, although they are remarkably close to the results of the similarity functional solution. We then discuss the fluctuation-dissipation theorem [12,13]. Section V generalizes our results for a non-Gaussian noise; Sec. VI contains the relevant computer simulations in the one-dimensional case. In Sec. VII we apply the same ideas to the two-dimensional case. Only a few cumulants are obtained explicitly but behavior similar to the Edwards-Wilkinson type is evident. There exists a possibility that above two dimensions the problem (1.1)–(1.3) becomes trivial. Numerical "roughening" exponents are briefly discussed.

II. THE HOPF EQUATION AND SIMILARITY FUNCTIONAL

We shall work with Eq. (1.2), and the field $h(\mathbf{x})$ is conventionally called the interface height profile [3]. Let $\mathcal{P}(h(\mathbf{x}),t)$ be the probability of finding the interface profile $h(\mathbf{x})$ at time t. It obeys the Fokker-Planck equation (Hopf equation) [9]

$$\partial_t \mathcal{P}(h(\mathbf{x}),t) = \int dx \left\{ D\frac{\delta^2}{\delta h(\mathbf{x})^2} - \frac{\delta}{\delta h(\mathbf{x})} \times [\nu \partial_x^2 h(\mathbf{x}) + \frac{\lambda}{2}[\partial_x h(\mathbf{x})]^2] \right\} \times \mathcal{P}(h(\mathbf{x}),t) . \quad (2.1)$$

The generation functional

$$\mathcal{G}(J(\mathbf{x}),t) = \int \mathcal{D}[h]\mathcal{P}(h(\mathbf{x}),t)\exp[\int J(\mathbf{x})h(\mathbf{x})dx] \quad (2.2)$$

satisfies the Fourier-transformed equation (2.1)

$$\partial_t \mathcal{G}(J(\mathbf{x}),t) = \int dx \left\{ DJ^2(\mathbf{x}) + J(\mathbf{x}) \left[\nu\partial_x^2 \frac{\delta}{\delta J(\mathbf{x})} + \frac{\lambda}{2}\left[\partial_x \frac{\delta}{\delta J(\mathbf{x})}\right]^2 \right] \right\} \mathcal{G}(J(\mathbf{x}),t) . \quad (2.3)$$

The logarithm of the solution $\mathcal{G}$ is assumed to be Taylor expandable:

$$\ln \mathcal{G} = F_1(t)\int J(\mathbf{x})dx + \tfrac{1}{2}\int J(\mathbf{x}_1)J(\mathbf{x}_2)F_2(\mathbf{x}_1-\mathbf{x}_2;t)dx_1dx_2 + \sum_{n=3}^{\infty}\frac{1}{n!}\int J(\mathbf{x}_1)\cdots J(\mathbf{x}_n)F_n(\mathbf{x}_1-\mathbf{x}_2,\ldots,\mathbf{x}_{n-1}-\mathbf{x}_n;t)dx_1\cdots dx_n , \quad (2.4)$$

where we used the translational invariance of the problem. The similarity functional ansatz is the assumption that for any n at length scales exceeding the cutoff a the cumulants are given by

$$F_n(\mathbf{x}_1-\mathbf{x}_2,\ldots,\mathbf{x}_{n-1}-\mathbf{x}_n) = L^{\alpha_n}(t)f_n\left[\frac{\mathbf{x}_1-\mathbf{x}_2}{L(t)},\ldots,\frac{\mathbf{x}_{n-1}-\mathbf{x}_n}{L(t)}\right] , \quad (2.5)$$

where all spatial dependences are scaled in terms of the correlation length

$$L(t) = t^{1/z} , \quad (2.6)$$

which is assumed to obey power-law behavior. Exponents α_n and z are to be determined together with the similarity functions f_n. Inserting expansion (2.4) into the Hopf equation we get the following hierarchy for F_n:

$$\frac{dF_1}{dt} = \frac{\lambda}{2!}\partial_{\mathbf{x}_1}\partial_{\mathbf{x}_2}F_2(\mathbf{x}_1-\mathbf{x}_2;t)|_{\mathbf{x}_1=\mathbf{x}_2} , \quad (2.7)$$

$$[\partial_t - \nu(\partial_{x_1}^2+\partial_{x_2}^2)]F_2(\mathbf{x}_1-\mathbf{x}_2;t) = 2D\delta(\mathbf{x}_1-\mathbf{x}_2) + \frac{\lambda}{3!}[\partial_{\mathbf{x}_1}\partial_{\mathbf{x}_3}F_3(\mathbf{x}_1-\mathbf{x}_2,\ldots,\mathbf{x}_2-\mathbf{x}_3;t)|_{\mathbf{x}_1=x_3} + \cdots] , \quad (2.8)$$

$$[\partial_t - \nu(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2)]F_3(\mathbf{x}_1-\mathbf{x}_2,\ldots,\mathbf{x}_2-\mathbf{x}_3;t) = \frac{\lambda}{4!}[\partial_{x_1}\partial_{x_4}F_4(\mathbf{x}_1-\mathbf{x}_2,\ldots,\mathbf{x}_3-\mathbf{x}_4;t)|_{\mathbf{x}_1=\mathbf{x}_4} + \cdots] + \lambda b_{22}\{\partial_{\mathbf{x}_1}F_2(\mathbf{x}_3-\mathbf{x}_1;t)\partial_{\mathbf{x}_2}F_2(\mathbf{x}_3-\mathbf{x}_2;t) + \cdots\} , \quad (2.9)$$

and so on, $b_{nm} = \frac{1}{2}[1/(n-1)!(m-1)!]$. The structure of higher-order equations can be seen from (2.7)–(2.9). In the equation of order n the notation $\cdots$ in square brackets stands for n possible pairings of the extra argument $\mathbf{x}_{n+1}$ with all $\mathbf{x}_1,\ldots,\mathbf{x}_n$. The notation $\cdots$ in curly brackets stands for $n!$ possible permutations of the indices *and* for all possible binary products of cumulants which result in the combined order $n+1$. By combined order we mean that, for example, the product $O(F_2F_2)$ entering Eq. (2.9) is of the fourth combined order. Note that the only inhomogeneity in the hierarchy is the noise-related δ function in Eq. (2.8). This is a distinct feature of Gaussian noise. More complex noises will be considered later.

It is important to avoid immediate substitution of the similarity ansatz (2.5) into the hierarchy, since (generally speaking) the nonlinear terms contain pairings of arguments, i.e., request cumulants at separations of its arguments where this ansatz is invalid. The problem here is twofold. First, we encounter distances of order a where continuous description is no longer applicable, and objects like $\delta(0)$ must be taken care of. Second, in the anomalous theories, pairing of arguments may generate new exponents. While the first part of the problem is resolved by introducing cutoff-related constants, the second part leads to new similarity forms for nonlinear terms and the hierarchy becomes underdetermined, with many possible solutions.

The situation simplifies in the case of nonanomalous theories; however there is in general little hope that the entire hierarchy can be solved as a whole. In our case, part of the solution of (2.7)–(2.9) has been available, and we found it possible to satisfy the hierarchy.

III. REPLICA BETHE ANSATZ

The solution of the replica Bethe ansatz by Kardar [6] will be rederived and used for discussing linear and time-independent terms in F_n's. For this purpose we apply the Hopf-Cole transformation [1,8] $h=(2\nu/\lambda)\ln w$ and find that w obeys the equation

$$\partial_t w=\nu\nabla^2 w+\frac{\lambda}{2\nu}\eta w \ , \qquad (3.1)$$

which is sometimes interpreted as the transfer-matrix equation for the Boltzmann weight of a directed polymer [13], when time plays the role of the longitudinal direction. Solving Eq. (3.1) and making the inverse transform we obtain

$$h(\mathbf{x},t)=\frac{2\nu}{\lambda}\ln\left\{\int dx_0\int_{\mathbf{x}(0)=\mathbf{x}_0}^{\mathbf{x}(t)=\mathbf{x}}\mathcal{D}[\mathbf{x}]\exp\left[-\int_0^t dt'\left[\frac{1}{4\nu}\left[\frac{d\mathbf{x}}{dt'}\right]^2+\frac{\lambda}{2\nu}\eta(\mathbf{x}(t'),t')\right]\right]\right\} . \qquad (3.2)$$

The beginnings of the trajectories entering (3.2) are uniformly distributed over the entire space to ensure $h(\mathbf{x},0)=0$. For convenience we recall the dimensions of the variables and parameters: $[\nu]=L^2/T$, $[D]=L^{d+2}/T$, $[\lambda]=L/T$, $[h]=L$, $[\eta]=L/T$, and $[w]=1$, where L and T stand for length and time, respectively.

In order to calculate different correlators of $h(\mathbf{x},t)$ in the method of replicas the logarithm entering (3.2) is replaced by $\ln w=(1/n)(w^n-1)|_{n\to 0}$, and the distribution (1.3) is employed to compute averages. One finds that if m trajectories $\mathbf{x}(t)$ intersect at a particular site $\mathbf{x},t$, the averaging gives

$$\int_{-\infty}^{\infty}\frac{d\eta\sqrt{a^d}}{\sqrt{4\pi D}}\exp\left[\frac{\lambda m\eta\theta}{2\nu}-\frac{\eta^2 a^d\theta}{4D}\right]=\exp\left[\frac{\lambda^2 m^2 D\theta}{4\nu^2 a^d}\right] , \qquad (3.3)$$

where a and θ are the spatial and temporal cutoffs, respectively. The problem is independent of the temporal cutoff. Note that the result (3.3) is model sensitive. Usage of different noise distributions may result in completely different replica interactions. This also stems from the fact that the spatial cutoff is explicitly present in the height correlators. The term m^2 is the signature of Gaussian noise; it indicates pair interactions of replicated trajectories. Kardar suggested writing it as $2[\frac{1}{2}m(m-1)+\frac{1}{2}m]$, i.e., decomposing (3.3) into linear weights and pair weights, the latter being proportional to the number of pairs $\frac{1}{2}m(m-1)$. Let us calculate the simplest correlator $\langle h(\mathbf{x},t)\rangle$ first. It is given by

$$\langle h(\mathbf{x},t)\rangle=\frac{2\nu}{\lambda}\lim_{n\to 0}[\langle w^n\rangle-1]$$

$$=\frac{2\nu}{\lambda}\frac{1}{n}\left[\prod_{j=1}^{j=n}\int dx_{0,j}\int_{\mathbf{x}_j(t)=\mathbf{x}_{0,j}}^{\mathbf{x}_j(t)=\mathbf{x}}\mathcal{D}[\mathbf{x}_j]\exp\left[-\int^t L_n dt'\right]-1\right]\Bigg|_{n\to 0} ,$$

$$L_n=\sum_{j=1}^{j=n}\left[\frac{1}{4\nu}\left[\frac{dx_j}{dt'}\right]^2+\frac{\lambda^2 D}{4\nu^2 a^d}\right]+\frac{\lambda^2 D}{2\nu^2}\sum_{i<j}\delta(\mathbf{x}_i-\mathbf{x}_j) , \qquad (3.4)$$

where L_n can be interpreted as the n-particle Lagrangian. Consider the n-particle wave function

$$\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)$$
$$=\prod_{j=1}^{j=n}\int dx_{0,j}\int_{\mathbf{x}_j(t)=\mathbf{x}_{0,j}}^{\mathbf{x}_j(t)=\mathbf{x}_j}\mathcal{D}[\mathbf{x}_j]\exp\left[\int^t L_n dt'\right] , \qquad (3.5)$$

which satisfies the Schrödinger equation

$$\partial_t\Psi=\hat{H}_n\Psi+\delta(t) , \qquad (3.6)$$

with the Hamiltonian

$$\hat{H}_n=\sum_{j=1}^{j=n}\left[\nu\frac{\partial^2}{\partial x_j^2}+\frac{\lambda^2 D}{4\nu^2 a^d}\right]+\frac{\lambda^2 D}{2\nu^2}\sum_{i<j}\delta(\mathbf{x}_i-\mathbf{x}_j) . \qquad (3.7)$$

The solution of Eq. (3.6) can be written as

$$\Psi=\sum_l e^{-E_{n,l}t}\psi_{n,l} , \tag{3.8}$$

where $E_{n,l}$ and $\psi_{n,l}$ are the eigenvalues and eigenfunctions of

$$\hat{H}_n\psi_{n,l}=-E_{n,l}\psi_{n,l} , \tag{3.9}$$

and summation in (3.8) is performed over all discrete and continuous states l. In the rest of Sec. III and throughout Sec. IV we shall limit ourselves to the *one*-dimensional case.

The ground state of the Hamiltonian (3.7) was found by Kardar [6], and it is given by the Bethe ansatz

$$\psi_{n,0}=\psi_0\exp\left[-\frac{\lambda^2 D}{2\nu^3}\sum_{i<j}|x_i-x_j|\right] , \tag{3.10}$$

with the energy

$$E_{n,0}=-\frac{\lambda^2 D}{4\nu^2 a}n-\frac{\lambda^4 D^2}{12\nu^5}n(n^2-1) . \tag{3.11}$$

Equations (3.10) and (3.11) can be verified by inspection. Naive generalizations of this ansatz to, say, two dimensions with K_0 modified Bessel functions instead of exponentials do not work. The wave function $\psi_{n,0}$ does not satisfy the initial condition $\psi=1$, so that excited states have to be invoked as discussed by Bouchaud and Orland [6]. These authors list two major problems with the replica Bethe-ansatz solution of our problem: (i) moments of w diverge too quickly while moments of h are well defined, and (ii) excited states grow in time more slowly than the ground state, while continuous states decay exponentially; this encourages one to take the limit $t\rightarrow\infty$ first and $n\rightarrow 0$ second. We shall see below that these problems [most probably (ii)] lead to losing terms. Given that the low excited states are included, calculations become quite difficult and additional model assumptions have been used by Bouchaud and Orland [6] to obtain the exponents of the correlation length. It is shown below how to get them directly from (3.10) and (3.11) without employing excited states.

The ground state is part of the solution and one may be interested in the part of the final average provided by this state. We then set the amplitude $\psi_0=1$ and find $\Psi(x,\ldots,x;t)=\exp(-E_n t)$, so that

$$\langle h(x,t)\rangle_0=\frac{2\nu}{\lambda}\lim_{n\rightarrow 0}\frac{\langle w^n\rangle-1}{n}=-\frac{2\nu}{\lambda}\lim_{n\rightarrow 0}\frac{E_{n,0}t}{n}=\left[\frac{\lambda D}{2\nu a}-\frac{\lambda^3 D^2}{6\nu^4}\right]t . \tag{3.12}$$

Similar calculations yield

$$\langle h(x_1,t)h(x_2,t)\rangle_0=\left[\frac{2\nu}{\lambda}\right]^2\lim_{n,m\rightarrow 0}\frac{\langle[w^n(x_1,t)-1][w^m(x_2,t)-1]\rangle}{nm}=-\frac{2D}{\nu}|x_1-x_2| , \tag{3.13}$$

$$\langle h(x_1,t)h(x_2,t)h(x_3,t)\rangle_0=\frac{2\lambda D}{3\nu^2}t , \tag{3.14}$$

and zero for all higher moments. The index 0 emphasizes that these results are obtained with the help of the ground state. Let us discuss them. First of all, Eq. (3.12) predicts the inversion of the growth direction with increase of λ at the point $\lambda=3\nu^3/aD$, while Eq. (1.2) averaged over space ensures that $\partial_t\langle h(x,t)\rangle$ is non-negative. This phenomenon is cutoff dependent and the second term in (3.12) should be dismissed in the continuous limit as compared to the first one in parentheses in the right-hand side of (3.12). At the same time this is a warning that the discrete version of the problem (1.1)–(1.3) may have properties different from the continuous version, and influence, for example, numerical studies at large λ. The possibility that the discrete model may have its own behavior was recognized by Kardar [6].

Equation (3.14) indicates that the celebrated exponent $h\propto t^{1/3}$ is present in the distribution. One then expects terms $t^{2/3}$ in (3.13) [14], which are absent. It is also clear that answers (3.12)–(3.14) are not actually the moments; they look closer to cumulants, since terms $\langle h\rangle^2$ are absent in (3.13), and so on. One may conclude that only the terms which are linear in time and independent of time are obtained from the ground state. Barring the incompleteness, the replica results happen to be already sufficient for determining basic exponents if one assumes global scaling of h fluctuations. The precise meaning of this global scaling is that the exponents α_n in (2.5) are proportional to n with the exception of α_1. Then, Eq. (3.13) gives $h\sim L^{1/2}$ and Eq. (3.14) results in $h\sim t^{1/3}$, so that $L\sim t^{2/3}$ is the correlation length. We shall see in the next section that expressions for cumulants of h are very close to (3.12)–(3.14). Despite the obvious success of the replica method, and help that we obtained from it in the next section, we feel that at the present stage of understanding it remains uncontrolled for our problem.

The nontrivial contribution to the third moment (3.14) means that the distribution of h at short length scales is *not* Gaussian. At first glance this contradicts the statement that the so-called fluctuation-dissipation theorem (FDT) applies to Burgers equation within the correlation length [12,13]. The reader may be interested in explicit checking of the FDT. It can be done with the help of Ref. [13], where the authors use scaling arguments and argue that the distribution

$$\mathcal{P}_0[h]\propto\exp\left[-\frac{\nu}{2D}\int(\nabla h)^2 dx\right] \tag{3.15}$$

(in our notation) holds at distances $x\ll L(t)$ in one dimension. It is a straightforward task to put (3.15) into the Hopf equation (2.3), assuming that the left-hand side vanishes, $\partial_t\mathcal{P}=0$, for such a steady state. Diffusive terms cancel,

$$\left[D\frac{\delta}{\delta h(\mathbf{x})}-\nu\nabla^2 h(\mathbf{x})\right]\mathcal{P}_0=0 , \tag{3.16}$$

and we are left with the integral

$$\int dx \nabla^2 h (\nabla h)^2 \ . \tag{3.17}$$

This integral is nonzero above one dimension and, therefore, the distribution (3.15) is not a solution. In one dimension the integral (3.17) is evaluated at the boundaries, and *may be* zero, although there is no guarantee so far. We recall that the boundaries of the integration region correspond to $x \sim L(t)$ and so far we know nothing about the distribution at such distances. Making the system periodic does not help either, provided that $L(t)$ does not exceed the system size.

We shall see in the next section that distribution (3.15) does describe *relative* fluctuations of the interface in one dimension, i.e., the fluctuations of the field $h(x,t) - h(x',t)$ for the separations $|x-x'|$ smaller than the correlation length. As for the *local* fluctuations of $h(x,t)$, they are not in a steady-state regime and cannot be described by (3.15). There is nothing novel in this separation of local and relative fluctuations—the same scenario takes place for the pure diffusion equation with external noise. However, in the latter case both are Gaussian.

We note in passing that the same distribution (3.15) may give the steady state of relative fluctuations for the stochastic partial differential equation of the type (1.2) with a quite general function of ∇h, including the function $\sqrt{1+(\nabla h)^2}$ [3].

IV. BUILDING THE SIMILARITY FUNCTIONAL

Let us understand the meaning of Eqs. (3.12)–(3.14) in terms of the hierarchy (2.7)–(2.9) and similarity ansatz (2.5). With Gaussian forcing the only source term enters Eq. (2.8), and it is appropriate to consider it first. If the correlator $\langle h(x_1,t)h(x_2,t)\rangle$ at late times or $|x_1-x_2| \ll L(t)$ contains the term $\propto |x_1-x_2|$ the diffusional operator in (2.8) gives a δ function which matches the source term. The same spatial dependence can be seen from (3.15). It is conceivable that in the second equation of the hierarchy the singularity (δ function) is balanced by the diffusion-based derivative of the second-order cumulant. Yet one could have used F_3 to balance the singularity, or both F_2 and F_3. However, we traced these extra possibilities further and found contradictions. Numerical studies [15] and the studies presented in Sec. VI also support Brownian-like behavior of the pair correlator at distances shorter than $L(t)$ with no contribution from higher cumulants.

The time-independent term in $\langle h(x_1,t)h(x_2,t)\rangle$ indicates that $\alpha_2 = 1$ and

$$F_2(x_1-x_2;t) = L(t) f_2\left[\frac{x_1-x_2}{L(t)}\right] = L(t)\left\{ f_2(0) + \frac{D}{\nu}\frac{|x_1-x_2|}{L(t)} + o\left[\frac{|x_1-x_2|}{L(t)}\right]\right\} , \tag{4.1}$$

where the expansion is applied at $|x_1-x_2| \ll L(t)$.

Returning to Eq. (2.7) we find that the term $\partial_{x_1}\partial_{x_2} F_2|_{x_1=x_2}$ is equal to $(D/\nu)\delta(0)$. This is the familiar object that must be of order of the inverse cutoff a^{-1}. The numerical coefficient remains undetermined, e.g., lattice version dependent. Equation (2.7) then gives

$$\frac{\alpha_1}{z} f_1 L^{\alpha_1 - z} \sim \frac{\lambda D}{2\nu}\delta(0) , \tag{4.2}$$

i.e.,

$$\alpha_1 = z, \quad f_1 \sim \lambda D / 2a\nu \ . \tag{4.3}$$

Thus the hierarchy is broken, and we shall build the rest of the similarity functional. Let us say a word of caution from the very beginning that the similarity solution thus obtained is not necessarily unique, and our nonanomalous solution does not, in principle, exclude possible anomalous ones. On the other hand, such a freedom allows one to make guesses.

Note that the usage of the similarity ansatz in Eq. (2.7) would result in the term

$$\partial_{x_1}\partial_{x_2} F_2|_{x_1=x_2} = L^{\alpha_2 - 2} f_2''(0) ,$$

i.e., provide us with incorrect scaling, $\alpha_1 - z = -1$. The discussed difference is induced by the singular δ function in Eq. (2.8), and is the only anomaly in the theory. Given that we balanced the δ function by F_2 (and not by F_3), we expect no additional singularities. The reason for this is that F_2 will only be differentiated once from now on; it happens in all the equations of the hierarchy starting from the fourth one. Each time it is balanced by the diffusion-based second derivative of the next order cumulant, which in its turn is "injected" further on in the form of its first derivative, and the singularity is completely healed. We are then ready to insert the similarity ansatz into the rest of the hierarchy.

Returning to Eq. (2.8) at finite distances and using the available relations among exponents, one finds

$$\frac{1}{z} L^{1-z}(t)[f_2(y_1-y_2) - (y_1-y_2) f_2'(y_1-y_2)] = \frac{\lambda}{3!} L^{\alpha_3-2}(t)[\partial_{y_1}\partial_{y_3} f_3(y_1,y_2,y_3)|_{y_1=y_3} + \partial_{y_2}\partial_{y_3} f_3(y_1,y_2,y_3)|_{y_2=y_3}] \tag{4.4}$$

which results in

$$z + \alpha_3 = 3 , \tag{4.5}$$

and at short distances

$$\partial_{y_1}\partial_{y_3} f_3(y_1,y_2,y_3)|_{y_1=y_3} + \partial_{y_2}\partial_{y_3} f_3(y_1,y_2,y_3)|_{y_2=y_3} = \frac{6}{z\lambda} f_2(0) \ . \tag{4.6}$$

Here we used abbreviated notation for arguments of f_3. Equation (4.6) shows that the third cumulant f_3 is probably the quadratic form of its arguments at short distances,

$$f_3=f_3(0)+\frac{2}{z\lambda}f_2(0)[(y_1-y_2)(y_1-y_3)+(y_2-y_1)(y_2-y_3)+(y_3-y_1)(y_3-y_2)] , \quad (4.7)$$

with $f_3(0)$ being the value of the cumulant with all three fields at the same point. Additional caution is to be exercized here: Eq. (4.7) is only one (maybe the simplest one) of many possible expressions which satisfy Eq. (4.6).

The next step is to go over to Eq. (2.9), which to the leading order takes the form

$$\frac{1}{z}L^{\alpha_3-z}(t)\left[\alpha_3 f_3(y_1,y_2,y_3)-\sum_{i<j}(y_i-y_j)\partial_{y_i-y_j}f_3(y_1,y_2,y_3)\right]$$
$$=\nu L^{\alpha_3-2}(t)(\partial^2_{y_1}+\partial^2_{y_2}+\partial^2_{y_3})f_3(y_1,y_2,y_3)+\frac{\lambda}{4!}L^{\alpha_4-2}(t)[\partial_{y_1}\partial_{y_4}f_4(y_1,\ldots,y_4)\Big|_{y_1=y_4}+\cdots]$$
$$+\lambda b_{22}\{\partial_{y_1}f_2(y_3-y_1;t)\partial_{y_2}f_2(y_3-y_2;t)+\cdots\} . \quad (4.8)$$

We now have to choose dominating time dependencies. If one assumes that α_3-2 is to contribute to the main order in Eq. (4.8), then the inequality $\alpha_3\geq 2$ follows from terms proportional to λ. Equation (4.5) gives $z\leq 1$ and we arrive at a contradiction, $\alpha_3-z>\alpha_3-2$, i.e., α_3-2 does *not* contribute to the leading order. One is left with three terms in (4.8) and four possible choices. Generally speaking, tracing all choices corresponds to an exponentially growing pattern of logical steps and may lead nowhere. We have found in our case that the only acceptable choice is to admit that all three remaining exponents are equal. It was also found that the same results for exponents are obtained if one makes use of the rule that the exponent of order n has to be determined from the equation of order n matching all lower-order λ-dependent terms. One finds $\alpha_3-z=0$ and

$$\alpha_1=\alpha_3=z=\tfrac{3}{2} . \quad (4.9)$$

Tracing higher orders one finds

$$\alpha_n=n/2, \quad n\geq 2 , \quad (4.10)$$

cf. [16]. The system of equations for the functions f_n remains unsolvable. However, the behavior of f_n at short distances can be extracted. Progress in this direction requires steps analogous to those made in deriving Eq. (4.7). Equation (4.8) becomes

$$f_3(0)=\frac{\lambda}{4!}[\partial_{y_1}\partial_{y_4}f_4(y_1,\ldots,y_4)|_{y_1=y_4}+\cdots]+\frac{\lambda D^2}{\nu^2} . \quad (4.11)$$

It is unclear from Eq. (4.11) if the derivatives of the fourth-order cumulant are nonzero. To find out the answer we have to go on to the next order

$$\tfrac{1}{3}[2f_4(y_1,y_2,y_3,y_4)-\sum_{i<j}(y_i-y_j)\partial_{y_i-y_j}f_4(y_1,y_2,y_3,y_4)]$$
$$=\frac{\lambda}{5!}[\partial_{y_1}\partial_{y_5}f_5(y_1,\ldots,y_5)|_{y_1=y_5}+\cdots]+\lambda b_{23}\{\partial_{y_1}f_2(y_3-y_1)\partial_{y_2}f_3(y_2,y_3,y_4)+\cdots\} . \quad (4.12)$$

It can be easily checked that the $O(f_2f_3)$ terms on the right-hand side are nonzero, and therefore f_4 or f_5 are nonzero. We again arrive at a logical branching point in building the similarity functional. If one assumes that f_4 is nonzero, then returning to Eq. (4.11) one finds a possible symmetric quadratic form for f_4,

$$f_4=f_4(0)+3\left[f_3(0)-\frac{\lambda D^2}{\nu^2}\right]\sum_{i<j}^{4}(y_i-y_j)^2 ; \quad (4.13)$$

the possible mixed form for f_5 is suggested by Eq. (4.12),

$$f_5=f_5(0)+12f_4(0)\sum_{i<j}^{5}(y_i-y_j)^2+O(y^3) , \quad (4.14)$$

and so forth. The latter result implies that high-order cumulants do not form a steady-state distribution at short distances, since in the full cumulant F_5 the quadratic terms (4.14) are multiplied by $L^{\alpha_5-2}(t)=t^{1/3}$, i.e., grow with time. Although interface width is insensitive to these corrections, one has *no* steady-state distribution at short distances. The only possibility to avoid this is to choose

$$f_3(0)=\frac{\lambda D^2}{\nu^2} . \quad (4.15)$$

We performed numerical simulations to extract the behavior of high-order local and nonlocal cumulants.

The results of these simulations are described in Sec. VI. We find that *relative* fluctuations are convincingly Gaussian at short length scales. Therefore the possibility (4.15) is realized, and f_4 is zero *together with* $f_4(0)$. It is the fifth-order cumulant which matches the $O(f_2 f_3)$ term in Eq. (4.12). This route leads to the Taylor expansion for high-order cumulants of odd orders

$$f_n = O(y^{(n+1)/2}) , \tag{4.16}$$

and the nonlocal part of the full cumulant F_n, which is proportional to $[L(t)]^{\alpha_n-(n+1)/2}=t^{-1/3}$, vanishes in the steady state. The local terms $f_n(0)$ for odd $n=5,7,\ldots$ are all zero. See the next section for further discussion of numerical results on high-order local cumulants.

At this stage it is appropriate to list the available answers for the one-dimensional problem. The cumulants in (2.4) are given by

$$\begin{aligned}
&F_1 \sim \frac{\lambda D t}{2a\nu} ,\\
&F_2(x_1-x_2)=t^{2/3}f_2((x_1-x_2)/t^{2/3})\\
&\qquad \to f_2(0)t^{2/3}+\frac{D}{\nu}|x_1-x_2| ,\\
&F_3(x_1,x_2,x_3)=tf_3((x_1-x_2)/t^{2/3},\ldots)\to \frac{\lambda D^2 t}{\nu^2} ,\\
&F_n(x_1,\ldots,x_n)=t^{n/3}f_n((x_1-x_2)/t^{2/3},\ldots)\to 0 ,
\end{aligned} \tag{4.17}$$

and the correlation length of the problem is $L(t)=t^{2/3}$. The expressions following the sign $\to$ are the nonvanishing asymptotic terms at late times (short distances), $|x_i-x_j| \ll t^{2/3}$ for all i,j. We found only one new term not contained in (3.12)–(3.14). Numerical factors do not match exactly, although we tried to eliminate possible errors. The results (4.17) support the assumption of global scaling used in the renormalization group method for late times. That is, if one disregards the uniform growth described by F_1, which can be compensated by adding a constant to the right-hand side of Eq. (1.2), and introduces the rescalings $x=\sigma x'$, $t=\sigma^{3/2}t'$, and $h=\sigma^{1/2}h'$ [13] then the results (4.17) do not change.

V. NON-GAUSSIAN NOISES

By a non-Gaussian noise we mean the general case when all cumulants of noise may be nonzero. However, the noise is still assumed to be δ-function correlated in space and time. Such a generalization can be accounted in the hierarchy (2.7)–(2.9) by adding the noise cumulants to the right-hand sides. Equations (2.7) and (2.8) are not modified, and, beginning with Eq. (2.9), we add $mD_m\delta(\mathbf{x}_1-\mathbf{x}_2)\cdots\delta(\mathbf{x}_1-\mathbf{x}_m)$ where D_m is the mth cumulant of the noise.

It is useful to collect the homogeneous relations for different exponents that we found in the previous section,

$$\begin{aligned}
&\alpha_1=z ,\\
&\alpha_2-z=\alpha_3-2>\alpha_2-2 ,\\
&\alpha_3-z=\alpha_4-2=\alpha_2+\alpha_2-2>\alpha_3-2 ,\\
&\alpha_4-z=\alpha_5-2=\alpha_2+\alpha_3-2>\alpha_4-2 ,
\end{aligned} \tag{5.1}$$

where we added inequalities featuring the dismissed diffusive terms. Our results in one dimension from the previous section are obtained by the inhomogeneous source at the level 2, where the $d=2$ Laplacian $(\partial_{x_1}^2+\partial_{x_2}^2)$ is balanced by the $d=1$ δ function $2D\delta(x_1-x_2)$. The solution satisfies the $d=1$ Laplace (or rather Poisson) equation, i.e., it is proportional to $|x_1-x_2|$ and $\alpha_2=1$. This reduction in dimensionality is due to the translational invariance. In the case of non-Gaussian noise we intend to consider separately the influence of mth cumulant and select the value of m which results in a set of the largest $\{\alpha_n\}$'s. Suppose that D_3 is the only nonzero cumulant. Then the noise $d=2$ δ function $3D_3\delta(x_1-x_2)\delta(x_1-x_3)$ is balanced by the $d=3$ Laplacian in Eq. (2.9). It has a $d=2$ logarithmic solution, and one concludes that $\alpha_3=0$. Solving the hierarchy (5.1) with $\alpha_3=0$ we discover the exponents $\alpha_1=z=2$, $\alpha_n=0$ and note that the diffusive terms cannot be neglected. Clearly, this set of $\{\alpha_n\}=0$ is negligible with respect to the set $\{n/2\}$ found in the previous section. Analogously, the fourth cumulant leads to a balance between the $d=4$ Laplacian and $d=3$ δ function. The resulting exponent $\alpha_4=-1$ leads to a solution of (5.1) which forms a negligible set of $\{\alpha_n\}$.

Thus, unless one deals with the rather special case of the zero second cumulant of the noise, the behavior outlined in the previous section becomes dominant in time. One can now relax the condition of δ correlation of the noise. It is clear that our results apply to all sufficiently rapidly decaying noise correlators in space and time provided that the large-x asymptotic of the solution of the Laplace equations is preserved. The case of power-law correlators requires special handling, and can be analyzed along the same lines. It would be of interest to make the comparison with the results of Medina *et al.* [3].

VI. NUMERICAL SIMULATION OF THE ONE-DIMENSIONAL CASE

We performed simple numerical simulations of Eq. (1.2) on Sun Sparc 2 computer workstation using the discrete scheme [17]

$$\begin{aligned}
h(x,t+\Delta t)=\;&h(x-1,t)+h(x+1,t)-2h(x,t)\\
&+\bar{\lambda}\{1-[1+g(x,t)/2]^{-1/2}\}+\eta(x,t)\sqrt{\Delta t} ,
\end{aligned} \tag{6.1}$$

$$g(x,t)=[h(x-1,t)-h(x,t)]^2+[h(x+1,t)-h(x,t)]^2 .$$

All parameters have been scaled into $\bar{\lambda}$. The stability introduced by this numerical scheme is rather robust and allowed us to use time steps $\Delta t=0.01$ without risk of running into divergences. The size of the h array with periodic boundary conditions was selected to be 500 lattice units and averaging over 1000 systems was performed. The numerical noise $\eta(x,t)$ was selected to be an unsophisticated random number generator of uniform noise on $[-1,1]$. The initial condition was selected to be $h(x,0)=0$. The typical run up to times $t=100$ takes a day on the Sun Sparc 2 workstation.

Figure 1 displays the time dependence of moments of the relative interface width

$$W_n(x,t)=\langle [h(x,t)-h(0,t)]^n\rangle^{1/n} \tag{6.2}$$

at short length scales $x=10$ lattice units as a function of time. The curves bend down at $t\sim 30$ and interface width saturates for $n=2$, 4, 6, 8, and 10. Thus the numerical steady state is achieved, and if we scale the widths by the factor $[(n-1)!!]^{1/2n}$ which represents the number of Gaussian pairs, we get that the curves are nicely superimposed, so that the distribution of relative fluctuations at short length scales is Gaussian. We believe that the remaining noise is due to insufficient averaging, in fact, the value $(W_4-W_2)/W_2$ is found to obey the root-mean-square deviation law.

The time dependence of local cumulants is shown in Fig. 2. The first cumulant is just the averaged interface

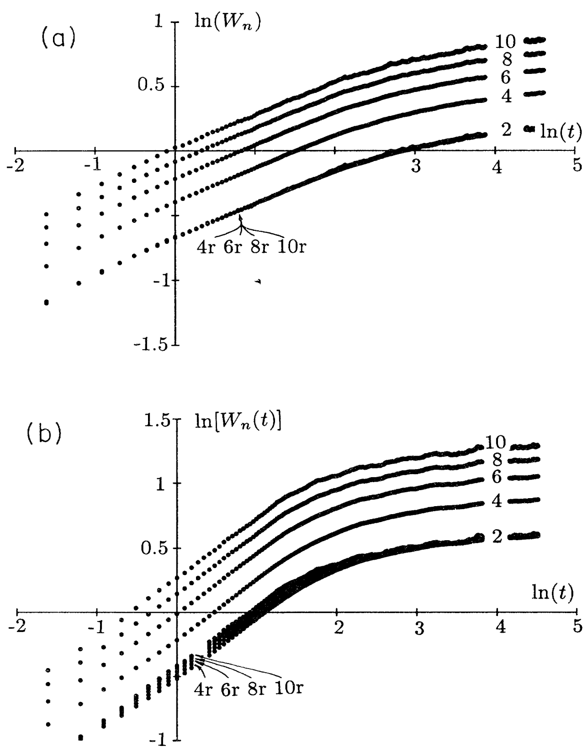

FIG. 1. Time dependence of the interface width moments W_n, Eq. (6.2). The curves are labeled with the numbers n of corresponding W_n. In case (a) $\bar{\lambda}=2$; (b) $\bar{\lambda}=10$. In case (a) the curve for W_2 is thick since the rescaled W_n (see text) collapses on top of it. In the case (b) it happens at late times, while initially the spatial distribution is clearly non-Gaussian. This is not a lattice effect.

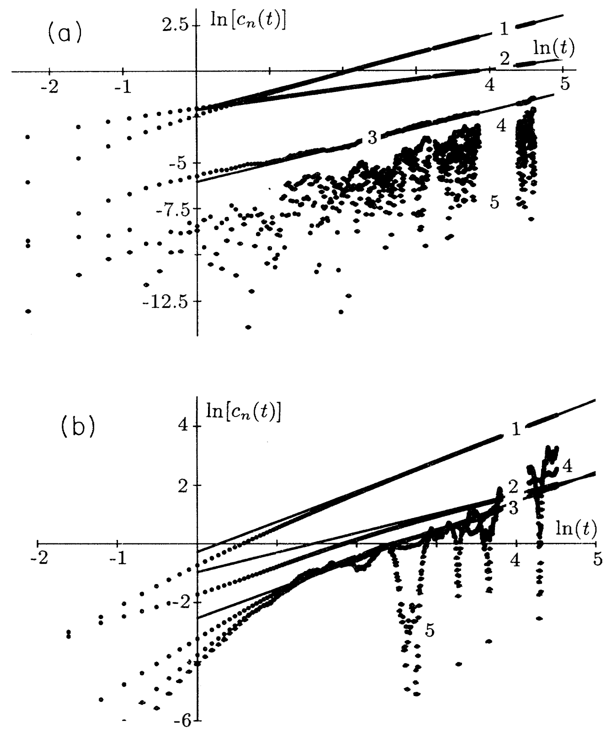

FIG. 2. Time dependence of local cumulants of the interface height. (a) $\bar{\lambda}=2$; (b) $\bar{\lambda}=10$. The straight lines are best fits, which give the numerical exponents for (a) $\alpha_1/z=1.01(1)$, $\alpha_2/z=0.65(1)$, $\alpha_3/z=0.97(5)$; and (b) $\alpha_1/z=1.02(1)$, $\alpha_2/z=0.68(1)$, $\alpha_3/z=1.01(5)$. Data for the fifth cumulant (where distinguishable) are shown with slightly elliptical dots.

height; the second and third are given by the formula

$$c_n(t)=\langle [h(x,t)-\langle h(x,t)\rangle]^n\rangle \; ; \tag{6.3}$$

the expressions for higher ones are a bit more cumbersome, and we refer the reader to mathematical textbooks. We see that for $\bar{\lambda}=2,10$ and $n=2,3$ the expected power laws develop after a transient, but are in good agreement with our expectations. The fourth and fifth *numerical* cumulants are, of course, nonzero, although their behavior is irregular in time at our level of averaging (1000 systems). Kim, Moore, and Bray [18] used a different numerical scheme and much more extensive averaging. They reported nonanomalous numerical exponents for nonzero high cumulants; see also [19] for a discussion. As is shown in the previous section the nonzero local high cumulants are incompatible with the steady state for relative fluctuations. Certainly, this argument does not necessarily apply to lattice models. The case of $\bar{\lambda}=10$ displays a new feature. Here the lattice effects are more "organized" and rare events associated with sign change of fluctuating cumulants are intermittent with steady growth of $f_{4,5}$. We consider this as an indication that at large enough $\bar{\lambda}$ the connection with the continuous equation is lost to a greater extent (rather than the establishing connection). Further numerical analysis seems to be appropriate to study the evolution of numerical high-order cumulants in different lattice models and the mentioned connection with the continuous case. This work is in progress.

VII. THE TWO-DIMENSIONAL CASE

Let us apply the same analytical method to $d=2$ case. The singularity in Eq. (2.8) leads to the term

$$\frac{D}{\nu}\ln\frac{|\mathbf{x}_1-\mathbf{x}_2|}{a} \tag{7.1}$$

for the second-order cumulant F_2. It is impossible to represent this expression as a part of a function of (2.5) type which depends exclusively on $|\mathbf{x}_1-\mathbf{x}_2|/L(t)$ and is multiplied by a power α_2 of $L(t)$. This complication is due to the presence of logarithms. Accounting for logarithms is a delicate issue which sometimes results in answers correct only with logarithmic precision. One has to decide first if the correlation length acquires logarithmic dependences. Our study of this problem with random initial conditions indicated that the correlation length does contain logarithms [20],

$$L(t)=t^{1/2}\ln^{1/2}(\nu t/a^2) . \tag{7.2}$$

We have also seen that in one dimension the correlation lengths are the same for both problems (noise-driven and random initial condition cases). This is our motivation to use (7.2) for the noise-driven case in two dimensions as well.

Then only minor modification is needed: the values $f_n(0)$ will be allowed to have logarithmic time dependences. The second cumulant acquires the form

$$\begin{aligned} F_2(\mathbf{x}_1-\mathbf{x}_2,t)&=L^{\alpha_2}(t)f_2\left[\frac{\mathbf{x}_1-\mathbf{x}_2}{L(t)}\right]\\ &=L^0(t)\left\{f_2(0)+\frac{D}{\nu}\ln\frac{|\mathbf{x}_1-\mathbf{x}_2|}{L(t)}\right.\\ &\qquad\left.+o\left[\frac{|\mathbf{x}_1-\mathbf{x}_2|}{L(t)}\right]\right\}, \end{aligned} \tag{7.3}$$

with $f_2(0)=c_1+c_2\ln L(t)$, $c_{1,2}$ cutoff-dependent constants which make it possible to account for (7.1), and $\alpha_2=0$. The first equation of the hierarchy again reduces to (4.2) and (4.3) with $\delta(0)\sim 1/a^2$, $\alpha_1=z$, and

$$F_1\sim\frac{\lambda Dt}{2\nu a^2} . \tag{7.4}$$

Note that the same arguments result in linear motion of the averaged interface height in higher dimensions (with a^2 to be replaced by a^d). Returning to Eq. (2.8) at finite separations one obtains the expression

$$\begin{aligned} c_2\frac{d\ln L(t)}{dt}=\frac{\lambda}{3!}L^{\alpha_3-2}(t)[&\partial_{y_1}\partial_{y_3}f_3(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)|_{\mathbf{y}_1=\mathbf{y}_3}\\ &+\partial_{y_2}\partial_{y_3}f_3(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)|_{\mathbf{y}_2=\mathbf{y}_3}] . \end{aligned} \tag{7.5}$$

Assuming that with logarithmic precision

$$d[\ln L(t)]/dt=1/zt=(1/z)L^{-z}(t)$$

one finds

$$z+\alpha_3=2 . \tag{7.6}$$

At short distances Eqs. (4.6) and (4.7) with two-dimensional arguments are valid. At the level of Eq. (4.11) the above mentioned convention leads to $\alpha_3=0$, and one finds from (7.6) $\alpha_1=z=2$. At this stage we have to reconsider the importance of the diffusive terms in the hierarchy. Tracing higher orders and using the same rule as in Sec. IV we find

$$\alpha_n=0 . \tag{7.7}$$

The obtained z exponent implies that $L(t)$ is proportional to $t^{1/2}$ times an arbitrary power of a logarithm, which we cannot fix by this method. We think that (7.2) is the correct answer. Returning to the level of power laws one may claim that the emerging similarity functional is essentially what is believed to be the Edwards-Wilkinson type of behavior [21]. However, the higher-order cumulants differ from their zero counterparts for the evolution of diffusion equation. For example, the fourth cumulant is nonzero; it satisfies an equation

$$\begin{aligned} f_3(0)=&\frac{\lambda}{4!}[\partial_{y_1}\partial_{y_4}f_4(\mathbf{y}_1,\ldots,\mathbf{y}_4)|_{\mathbf{y}_1=\mathbf{y}_4}+\cdots]\\ &+\frac{\lambda D^2}{3\nu^2}\left\{\frac{(\mathbf{y}_3-\mathbf{y}_1)(\mathbf{y}_3-\mathbf{y}_2)}{|\mathbf{y}_3-\mathbf{y}_1||\mathbf{y}_3-\mathbf{y}_2|}+\cdots\right\}, \end{aligned} \tag{7.8}$$

with two other permutations of the indexes in curly brackets. There is no possibility to absorb the inhomogeneity into $f_3(0)$ as we managed to do in one dimension.

The power-law renormalization group (RG) rescaling of our results $x=\sigma x'$, $t=\sigma^2 t'$, $h=h'$ is consistent (at the level of power laws) with the available results of Edwards-Wilkinson type [21], although the distribution is "logarithmic to all orders." We do not know whether the Burgers turbulence is renormalizable since there is no action in this model, and RG methods (if any) must be somehow modified. As in the one-dimensional case, the general non-Gaussian noise does not lead to a different behavior of the solution found.

The appearance of the logarithmic correlator in $d=2$, see Eq. (7.3), is the signature of the transition to the new regime [22]. As we mentioned, when $z=2$ the diffusional terms can no longer be dismissed; more than that, they begin to dominate above two dimensions. Thus the nonlinear term is no longer significant, and results can be inferred from the simple diffusion equation.

We now discuss the diverse numerical results [15] on interface roughness in two dimensions which reported different exponents of interface width, $W(t)\propto t^{\beta}$, in the range $0\le\beta\le 0.25$. Numerical schemes and different solid-on-solid models correspond to different discretizations of Eq. (1.2). "Strong-coupling" results are usually obtained with large nonlinearity or strong noise amplitude. The problem in this regime is known to be sensitive to lattice formulation or noise distribution and even has a remarkable phase transition with the increase of disorder [23]. For instance, the transition observed numerically by Amar and Family [15] may be the indication that the behavior of the discretized scheme is no longer relevant to the continuous Eqs. (1.1)–(1.3) with Gaussian noise. These results are perfectly relevant to their discrete model. At the moment this discussion expresses just a possible viewpoint.

VIII. CONCLUSION

It seems possible that other field-theoretical models can be treated by this method. It is not clear in advance that the hierarchy can be analyzed even if one breaks it by using the correct similarity ansatz introduced in accordance with symmetry requirements or numerical results. The numerical results are sometimes difficult to obtain; they are usually quite undetermined even in two dimensions. Attempts to minimize the number of relevant terms at each level of the hierarchy lead to exponentially growing logical schemes with more than one possible solution. When more than one similarity solution exists it is unclear which is actually realized in agreement with initial and/or boundary conditions. Given that the Hopf equation (Fokker-Planck equation) is linear the stability of the similarity functional does not depend upon the functional and therefore cannot be used for discrimination between similarity solutions. By analogy with linear partial differential equations, one may think that properly prepared sums over possible (similarity) solutions should meet the initial and boundary conditions which is, in general, a nontrivial problem.

The results of Secs. II–V indicate that the one-dimensional problem has scaling at late times which meets the requirements of the renormalization group method, notwithstanding the internal contradictions associated with actual application of this method [24]. In two dimensions (see Sec. VI) fields also scale, and from the renormalization group point of view the available solution must be placed into the "universality class" of pure diffusion equation.

Comparing the results of this paper with papers I and II we found that the relaxations of random initial conditions and noise-driven dynamics are sometimes characterized by equal exponents if distributions of the random fields which are used for the initial conditions and the permanent forcing are the same. For Gaussian distributions in one dimension the kinetic energy $(\partial h/\partial x)^2$ decays as $t^{-2/3}$, so that interface height grows as $h \sim t^{1/3}$. The same result is seen from (4.17) for the fluctuating part of the noise-driven problem. The correlation length dependence $L(t)=t^{2/3}$ is the same for both problems as we have mentioned. In two dimensions we found it self-consistent to have $L(t)=t^{1/2}\ln^{1/2}(\nu t/a^2)$ in both problems, and the interface height $h \sim \ln^{1/2}(t)$ is again the same in both cases [compare (7.5) and Eq. (8.6) of Ref. [8], paper II]. We consider this equivalence as a very strong support of the results presented above.

Equally consistent with the findings of sensitivity to random initial conditions presented in I and II are our present expectations that different noise distributions may lead to principally different behavior, including the phase transition [23]. The presence of sensitivity was also obtained by Sinai and by She, Aurell, and Frisch for fractional Brownian random initial conditions [25]. The application of all these ideas to the noise-driven case is the subject of future work.

We conclude this paper by taking a step back and recalling that there may still exist other solutions in the noise-driven case which are not realizable with analogous random initial conditions.

ACKNOWLEDGMENTS

I am indebted to T. J. Newman for his continuous attention to and interest in this problem and help. I am thankful to Nigel Goldenfeld for attracting my attention to the shortcomings of renormalization group and methods of similarity solutions, to Paul Wiegmann for helpful conversations, to Boris Spivak for sharing his results prior to publication and many clarifying and dramatic discussions, and to Mehran Kardar for the references connected with numerical high-order cumulants. This work was supported in part by the Material Research Laboratories at the University of Illinois at Urbana-Champaign and at the University of Chicago, and in part by NSF Grant No. NSF-DMR-89-20538.

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